Solution 8

Supplementary Problems

1. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be on $[a,b]$ and $[\alpha,\beta]$ respectively that describe the same curve C. It has been shown that there exists some φ maps $[a,b]$ one-to-one onto $[\alpha,\beta]$, $\varphi'(t)>0$, such that $\mathbf{r}_2(\varphi(t))=\mathbf{r}_1(t)$ when both parametrization runs in the same direction. When they runs in different direction, $\varphi'(t)<0$. Using this fact to prove that in both cases,

$$\int_a^b f(\mathbf{r}_1(t))|\mathbf{r}_1'(t)|\,dt=\int_\alpha^\beta f(\mathbf{r}_2(z))|\mathbf{r}_2'(z)|\,dz\ .$$

In other words, the line integral

$$\int_C f\,ds$$

is independent of the choice of parametrization with the same or opposite direction.

Solution. Differentiating the relation $\mathbf{r}_2(\varphi(t))=\mathbf{r}_1(t)$ and using the chain rule, we get

$$\mathbf{r}_2'(z)\varphi'(t)=\mathbf{r}_1'(t)\ ,$$

so

$$|\mathbf{r}_2'(z)||\varphi'(t)|=|\mathbf{r}_1'(t)|\ ,\quad z=\varphi(t).$$

We have

$$\begin{aligned}\int_{\mathbf{r}_2} f\,ds &= \int_\alpha^\beta f(\mathbf{r}_2(z))|\mathbf{r}_2'(z)|\,dz\\ &= \int_\alpha^\beta f(\mathbf{r}_2(z))\frac{|\mathbf{r}_1'(t)|}{|\varphi'(t)|}\,dz\ .\end{aligned}$$

When $\varphi(a)=\alpha,\varphi(b)=\beta$ and $\varphi'>0$, by the change of variables formula, we continue

$$\begin{aligned}&= \int_a^b f(\mathbf{r}_2(\varphi(t)))\frac{|\mathbf{r}_1'(t)|}{|\varphi'(t)|}\varphi'(t)\,dt\\ &= \int_a^b f(\mathbf{r}_1(t))|\mathbf{r}_1'(t)|\,dt\\ &= \int_{\mathbf{r}_1} f\,ds\ .\end{aligned}$$

On the other hand, when $\varphi(a)=\beta,\varphi(b)=\alpha$ and $\varphi'<0$, we have

$$\begin{aligned}&= \int_b^a f(\mathbf{r}_2(\varphi(t)))\frac{|\mathbf{r}_1'(t)|}{-\varphi'(t)}\varphi'(t)\,dt\\ &= \int_a^b f(\mathbf{r}_1(t))|\mathbf{r}_1'(t)|\,dt\\ &= \int_{\mathbf{r}_1} f\,ds\ .\end{aligned}$$

The same result holds.